Chapter 11. Radiation

How accelerating charges and changing currents produce electromagnetic waves, how they radiate.

11.2. Point charges: Power Radiated by a Moving Point Charge

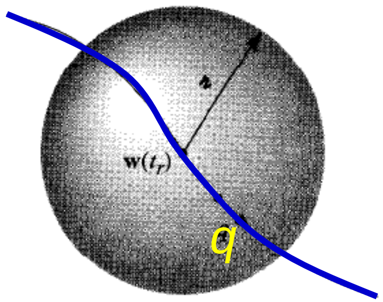
The fields of a point charge *q* in arbitrary motion is (Eq. 10.65)

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r},t)$$

The magnetic field of a point charge is always perpendicular to the electric field.

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2-v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \quad \text{where } \mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

velocity field (nonradiation field?) — acceleration field (radiation field?)



The Poynting vector is ➔ $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0 c}[\mathbf{E}\times(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E})] = \frac{1}{\mu_0 c}[E^2\hat{\boldsymbol{\mathscr{r}}} - (\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{E})\mathbf{E}]$

Consider a huge sphere of radius $\mathscr{r}$, the area of the sphere is proportional to $\mathscr{r}^2$

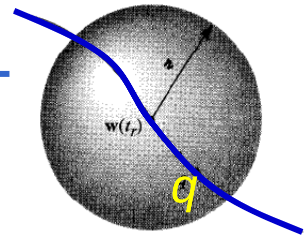
So any term in S that goes like $1/\mathscr{r}^2$ will yield a finite answer, but terms like $1/\mathscr{r}^3$ or $1/\mathscr{r}^4$ will contribute nothing in the limit $\mathscr{r} \to \infty$.

The velocity fields carry energy as the charge moves this energy is dragged along, **but it's not *radiation*.**

Only the *acceleration* fields represent true radiation (hence their other name, radiation fields):

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \qquad \mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}E_{\text{rad}}^2\hat{\boldsymbol{\mathscr{r}}}$$

Power Radiated by a Point Charge



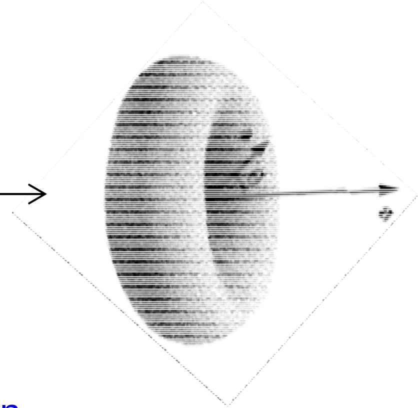
$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \qquad \mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}E^2_{\text{rad}}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \text{where } \mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

If the charge is instantaneously at *rest* (at time t_r), then $\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}}$, ***(It is good approximation as long as v « c.)***

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2 \mathscr{r}}[\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a})] = \frac{\mu_0 q}{4\pi\mathscr{r}}[(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a}]$$

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi\mathscr{r}}\right)^2[a^2 - (\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})^2]\hat{\boldsymbol{\mathscr{r}}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\boldsymbol{\mathscr{r}}}$$



where θ is the angle between $\hat{\boldsymbol{\mathscr{r}}}$ and $\mathbf{a}$.

No power is radiated in the forward or backward direction-rather,
it is emitted in a donut about the direction of instantaneous acceleration.

The total power radiated is

$$P = \oint \mathbf{S}_{\text{rad}}\cdot d\mathbf{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int \frac{\sin^2\theta}{\mathscr{r}^2}\mathscr{r}^2 \sin\theta\, d\theta\, d\phi \;\Rightarrow\; \boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}}$$

Larmor formula

An exact treatment of the case $v \neq 0$ is more difficult. Let's simply quote the result:

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\mathbf{v}\times\mathbf{a}}{c}\right|^2\right) \quad \text{where } \gamma \equiv 1/\sqrt{1 - v^2/c^2}.$$

Liénard's generalization of the Larmor formula

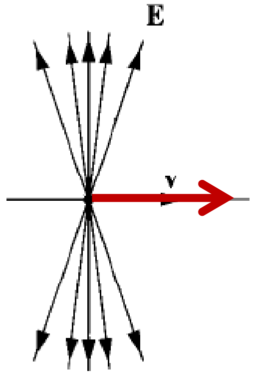
➔ ***The factor γ^6 means that the radiated power increases enormously as the velocity approaches the speed of light.***

Comparison: Non-radiated fields and radiated fields from a Point Charge

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \quad \text{where } \mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

$\mathbf{E}_{non\text{-}rad}$ velocity field (non-radiation field) acceleration field (radiation field) $\mathbf{E}_{rad}$

Note that the velocity fields also do carry energy; they just don't transport it out to infinity.

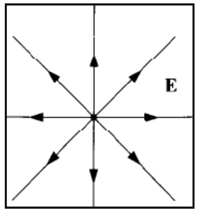


$a = 0, \upsilon \neq 0$

$$E_{rad} = 0 \qquad E_{non-rad} = \frac{q}{4\pi\epsilon_0} \frac{(c^2 - v^2)\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}\mathbf{u} = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta / c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}$$

$a = 0, \upsilon = 0$

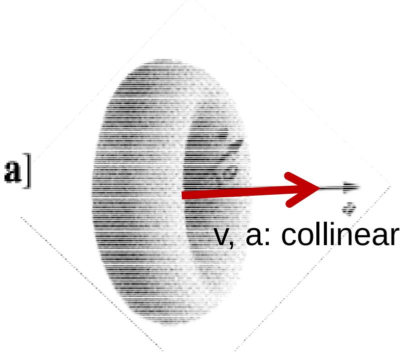
$$E_{rad} = 0 \qquad E_{non-rad} = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$



$a \neq 0, \upsilon \neq 0;\ \upsilon \ll c$

$$E_{non-rad} \neq 0 \qquad E_{rad} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[\boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] = \frac{\mu_0 q}{4\pi\mathscr{r}}[(\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a}]$$

$$\mathbf{S}_{\text{rad}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\boldsymbol{\mathscr{r}}} \qquad P = \oint \mathbf{S}_{\text{rad}} \cdot d\mathbf{a} = \frac{\mu_0 q^2 a^2}{6\pi c}$$

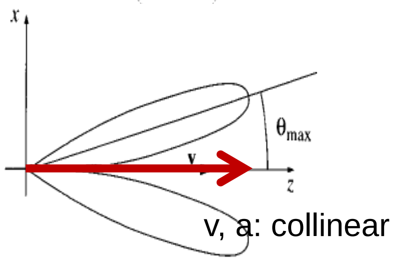


$a \neq 0, \upsilon \neq 0; \upsilon \sim c$

$$E_{non-rad} \neq 0 \qquad E_{rad} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[\boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})]$$

$$P = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c} \qquad \frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}$$

$$\beta \equiv v/c \qquad \gamma \equiv 1/\sqrt{1 - v^2/c^2}$$



11.2.2 Radiation Reaction

Radiation from an accelerating charge carries off energy → resulting in reduction of the particle's kinetic energy.
→ Under a given force, therefore, a charged particle accelerates *less* than a neutral one of the same mass.
→ The radiation evidently exerts a force (**F**$_{rad}$) back on the charge – *recoil (or, radiation reaction) force*.

For a nonrelativistic particle *(v* « c) the total power radiated is given by the Larmor formula (Eq. 11.70):

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \longrightarrow \mathbf{F}_{\mathrm{rad}} \cdot \mathbf{v} = -\frac{\mu_0 q^2 a^2}{6\pi c} \qquad (11.77)$$

Conservation of energy suggests that this is also the rate at which the particle loses energy, under the influence of the radiation reaction force **F**$_{rad}$:

The energy lost by the particle in any given time interval: $\int_{t_1}^{t_2} \mathbf{F}_{\mathrm{rad}} \cdot \mathbf{v}\, dt = -\frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} a^2\, dt$

$$\int_{t_1}^{t_2} a^2\, dt = \int_{t_1}^{t_2} \left(\frac{d\mathbf{v}}{dt}\right) \cdot \left(\frac{d\mathbf{v}}{dt}\right) dt = \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d^2\mathbf{v}}{dt^2} \cdot \mathbf{v}\, dt$$

0

If the motion is periodic-the velocities and accelerations are identical at t_1 and t_2,
or if $v \cdot a = 0$ at t_1 and t_2,

$$\int_{t_1}^{t_2} \left(\mathbf{F}_{\mathrm{rad}} - \frac{\mu_0 q^2}{6\pi c} \dot{\mathbf{a}}\right) \cdot \mathbf{v}\, dt = 0 \longrightarrow \mathbf{F}_{\mathrm{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\mathbf{a}}$$

Abraham-Lorentz formula
for the radiation reaction force

Radiation Reaction

$$\int_{t_1}^{t_2}\left(\mathbf{F}_{\text{rad}} - \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}\right)\cdot\mathbf{v}\,dt = 0 \qquad \mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}$$

Abraham-Lorentz formula
for the radiation reaction force

For suppose a particle is subject to no *external* forces (F = 0); then Newton's second law says

$$F_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{a} = ma \longrightarrow a(t) = a_0 e^{t/\tau}, \quad \text{where} \quad \tau \equiv \frac{\mu_0 q^2}{6\pi mc}$$

➔ In the case of the electron, $\tau = 6 \times 10^{-24}$ s. → only the time taken for light to travel ~ 10^{-15} m

➔ The acceleration spontaneously *increases* exponentially with time!
→ "runaway" under no external force!

If you *do* apply an external force,

$$ma = F_{rad} + F, \quad F_{rad} = \tau\dot{a} \quad \Rightarrow \quad a = \tau\dot{a} + \frac{F}{m}$$

: Abraham-Lorentz equation of motion

If an external force is applied to the particle for times t > 0, the equation of motion predicts "preaccelaeration" before the force is actually applied. ➔ It starts to respond *before the force acts!*
→ "*preacceleration*" *acausalty!*

(Problem 11.19) Assume that a particle is subjected to a constant force *F*, beginning at time *t* = 0 and lasting until time *T*. Show that you can *either* eliminate the runaway in region (iii) *or* avoid preacceleration in region (i), *but not both.*

Radiation Reaction

Problem 11.19 If you apply an external force, **F**, acting on the particle, Newton's second law for a charged particle becomes

$$\boxed{a = \tau \dot{a} + \frac{F}{m}} \qquad \tau \equiv \frac{\mu_0 q^2}{6\pi mc}$$

(b) A particle is subjected to a constant force *F,* beginning at time $t = 0$ and lasting until time *T.*
Find the most general solution *a (t)* to the equation of motion in each of the three periods: (i) $t < 0$; (ii) $0 < t < T$; (iii) $t > T$.

$$\text{(i) } a = \tau\dot{a} = \tau\frac{da}{dt} \Rightarrow \frac{da}{a} = \frac{1}{\tau}dt \Rightarrow \int\frac{da}{a} = \frac{1}{\tau}\int dt \Rightarrow \ln a = \frac{t}{\tau} + \text{constant} \Rightarrow \boxed{a(t) = Ae^{t/\tau},}$$

$$\text{(ii) } a = \tau\dot{a} + \frac{F}{m} \Rightarrow \tau\frac{da}{dt} = a - \frac{F}{m} \Rightarrow \frac{da}{a - F/m} = \frac{1}{\tau}dt \Rightarrow \ln(a - F/m) = \frac{t}{\tau} + \text{constant} \Rightarrow a - \frac{F}{m} = Be^{t/\tau} \Rightarrow \boxed{a(t) = \frac{F}{m} + Be^{t/\tau},}$$

$$\text{(iii) Same as (i): } \boxed{a(t) = Ce^{t/\tau},}$$

(c) Impose the continuity condition (a) at $t = 0$ and $t = T$.
Show that you can *either* eliminate the runaway in region (iii) *or* avoid preacceleration in region (i), *but not both.*

At $t = 0$, $A = F/m + B$; at $t = T$, $F/m + Be^{T/\tau} = Ce^{T/\tau} \Rightarrow C = (F/m)e^{-T/\tau} + B$.

$$a(t) = \begin{cases} [(F/m) + B]\,e^{t/\tau}, & t \le 0; \\ \left[(F/m) + Be^{t/\tau}\right], & 0 \le t \le T; \\ \left[(F/m)e^{-T/\tau} + B\right]e^{t/\tau}, & t \ge T. \end{cases}$$

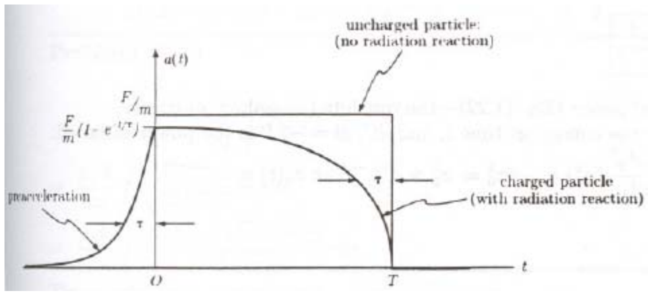
To eliminate the runaway in region (iii), we'd need $B = -(F/m)e^{-T/\tau}$;

to avoid preacceleration in region (i), we'd need $B = -(F/m)$.

⇨ Obviously, we cannot do both at once.

(d) If you choose to eliminate the runaway,

$$a(t) = \begin{cases} (F/m)\left[1 - e^{-T/\tau}\right]e^{t/\tau}, & t \le 0; \\ (F/m)\left[1 - e^{(t-T)/\tau}\right], & 0 \le t \le T; \\ 0, & t \ge T. \end{cases}$$



Radiation Reaction (J. Jacksons, p.780)

To estimate the range of parameters where radiative effects on reaction are important or not, consider the radiative energy of a charge ***e*** under an external force to have acceleration ***a*** for a period of time T:

For a particle at rest initially a typical energy is its kinetic energy after the period of acceleration:

$$E_0 \sim m(aT)^2$$

On the other hand, From the Larmor formula the energy radiated is of the order of

$$E_{rad} \sim P \cdot T \sim \frac{\mu_0 e^2}{6\pi c} a^2 T \quad \longleftarrow \quad P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

The criterion for the regime where radiative effects are not important can thus be expressed by

$$E_{rad} \ll E_0 \longrightarrow \frac{\mu_0 e^2}{6\pi c} a^2 T \ll m a^2 T^2 \longrightarrow T \gg \frac{\mu_0 e^2}{6\pi mc} = \tau$$

➔ In the case of the electron, $\tau = 6 \times 10^{-24}$ s. → only the time taken for light to travel ~ 10^{-15} m

➔ *Only for phenomena involving such distances or times will we expect radiative effects to play a crucial role.*

(ex) If the motion is quasi-periodic with a typical amplitude ***d*** and characteristic frequency ω_0: $E_0 \sim m\omega_0^2 d^2$

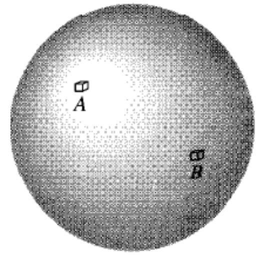
The acceleration are typically a ~ ω_0^2 d, and the time interval T ~ ($1/\omega_0$) :

$$E_{rad} \ll E_0 \longrightarrow \frac{\mu_0 e^2}{6\pi c} \frac{(\omega_0^2 d)^2}{\omega_0} \ll m\omega_0^2 d^2 \longrightarrow \frac{1}{\omega_0} \sim T \gg \tau$$

➔ *If the mechanical time interval is much longer than τ, radiative reaction effects will be unimportant.*

11.2.3 The Physical Basis of the Radiation Reaction

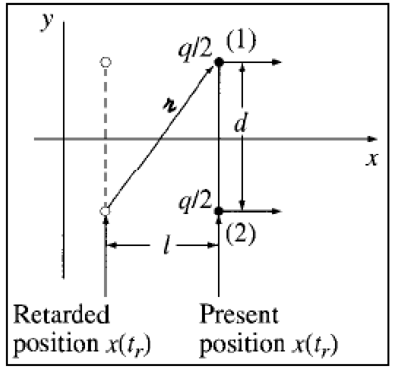
Conclusion: ***"****The radiation reaction is due to the force of the charge on itself ("****self-force****").*
Or, more elaborately, the net force exerted by the fields generated by different parts of the charge distribution acting on one another."



Consider a moving charge with an *extended* charge distribution:
In general, the electromagnetic force of one part (A) on another part (B) is not equal and opposite to the force of B on A.

Let's simplify the situation into a "bumble" : the total charge **q** is divided into two halves separated by a fixed **d**:

➔ In the point limit *(d* → 0), it must yield the Abraham-Lorentz formula.



The electric field at (1) due to (2) is

$$\mathbf{E}_1 = \frac{(q/2)}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u}-(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}] \qquad \mathbf{u} = c\,\hat{\boldsymbol{\mathscr{r}}} \text{ and } \boldsymbol{\mathscr{r}} = l\,\hat{\mathbf{x}} + d\,\hat{\mathbf{y}}$$

$$u_x = \frac{cl}{\mathscr{r}} \longrightarrow E_{1_x} = \frac{q}{8\pi\epsilon_0 c^2}\frac{(lc^2-ad^2)}{(l^2+d^2)^{3/2}} \quad \text{By symmetry, } E_{2_x} = E_{1_x},$$

$$\mathbf{F}_{\text{self}} = \frac{q}{2}(\mathbf{E}_1+\mathbf{E}_2) = \frac{q^2}{8\pi\epsilon_0 c^2}\frac{(lc^2-ad^2)}{(l^2+d^2)^{3/2}}\hat{\mathbf{x}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a(t)}{4c^2 d}+\frac{\dot{a}(t)}{3c^3}+(\ \ )d+\cdots\right]\hat{\mathbf{x}}$$

The first term ~ E_0

The second term survives in the "point dumbbell" limit *d* → 0: $F_{\text{rad}}^{\text{int}} = \frac{\mu_0 q^2 \dot{a}}{12\pi c}$

This term (x 2) is equal to the radiation reaction force given by the Abraham-Lorentz formula!

➔ ***In conclusion,*** ***"****the radiation reaction is due to the force of the charge on itself ("****self-force****").*