Paper PT-204 UNIT - III

Lagrange Interpolation

1. Statement and applications

Consider the following table:

x	x_0	x_1	x_2	x_3	….	xk	….	x_n
f(x)	f_0	f_1	f_2	f_3	…	f_k	….	f_n

In the above table, f_k, k = 0, ·· . , n are assumed to be the values of a certain function f(x), evaluated at x_k, k = 0, ··· , n in containing these points. Here it is important that only the functional values (f_k) are known, not the function f(x) itself. The problem is to find f_u corresponding to a **nontabulated** intermediate value x = u.

Such type of problems is known as Interpolation Problem. The independent values of the function x_0, x_1, ··· , x_n are called the nodes.

In short, we can state that Given (n + 1) points: (x_0, f_0), (x_1, f_1), ···(x_n, f_n), find f_u corresponding to x_u, where $x_0 < x_u < x_n$; assuming that f0, f1, ··· , fn are the values of a certain function f(x) at x = x_0, x_1, ··· , x_n, respectively.

The Interpolation problem is also a classical problem and dates back to the time of Newton and Kepler, who needed to solve such a problem in analyzing data on the positions of stars and planets. It is also of interest in numerous other practical applications. Here is an example.

2 Existence and uniqueness

It is well-known that a continuous function $f(x)$ on $[a, b]$ can be **approximated** as close as possible by means of a polynomial. Specifically, for each $\varepsilon > 0$, there exists a polynomial $P(x)$ such that $|f(x) - P(x)| < \varepsilon$ for all x in $[a, b]$.

This is a classical result, known as **Weierstrass Approximation Theorem**.

Knowing that f_k, $k = 0, \cdots, n$ are the values of a certain function at x_k, the most obvious thing then to do is to construct a polynomial $P_n(x)$ of degree at most n that passes through the $(n + 1)$ points: (x_0, f_0), (x_1, f_1), $\cdots$, (x_n, f_n).

Indeed, if the nodes $x_0, x_1, ..., x_n$ are assumed to be distinct, then such a polynomial always does exist and is unique, as can be seen from the following.

Let $P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial of n^{th} degree. If $P_n(x)$ interpolates at $x_0, x_1, \cdots, x_n$, we must have, by definition

$$P_n(x_0) = a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_nx_0^n = f_0$$
$$P_n(x_1) = a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_nx_1^n = f_1$$
$$\cdots$$
$$\cdots \qquad (1)$$
$$P_n(x_n) = a_0 + a_1x_n + a_2x_n^2 + \cdots + a_nx_n^n = f_n$$

These equations can be written in matrix form:

$$\begin{bmatrix} 1 & x0 & .. & x0^n \\ 1 & x1 & .. & x1^n \\ - & - & .. & .. \\ 1 & xn & .. & xn^n \end{bmatrix} - \begin{pmatrix} a0 \\ a1 \\ \\ an \end{pmatrix} = \begin{pmatrix} f0 \\ f1 \\ \\ fn \end{pmatrix} \qquad (2)$$

Because $x_0, x_1, \cdots, x_n$ are distinct, it can be shown (with the help of data) that the matrix of the above equation (2) is nonsingular. Thus, the linear system for the unknowns $a_0, a_1, \cdots, a_n$ has a unique solution, in view of the following well-known result.

The n × n algebraic linear system Ax = b has a unique solution for every b if and only if A is nonsingular.

This means that $P_n(x)$ exists and is unique.

It can be summarize as follows:

Given (n + 1) distinct points $x_0, x_1, \cdots, x_n$ and the associated values of a function f(x) at these points (i.e., $f(x_i) = f_i$, $i = 0, 1, \cdots, n$), there is a **unique polynomial $P_n(x)$** of at most n^{th} degree such that $P_n(x_i) = f_i$, $i = 0, 1, \cdots, n$. The coefficients of this polynomial can be obtained by solving the (n + 1) ×(n + 1) linear system using either Gauss elimination method or Gauss Seidal Method.

The polynomial $P_n(x)$ is called the interpolating polynomial.

3 The Lagrange Interpolation

Once we know that the interpolating polynomial exists and is unique, the problem then becomes how to construct an interpolating polynomial; that is, how to construct a polynomial $P_n(x)$, such that

$P_n(x_i) = f_i$, $i = 0, 1, \cdots, n$.

It is natural to obtain the polynomial by solving the linear system (equation (1)) in the previous section. Unfortunately, the matrix of this linear system, known as the **Vandermonde Matrix**, is usually highly ill-conditioned, and the solution of such an ill-conditioned system, even by the use of a stable method, may not be accurate. There are, however, several other ways to construct such a polynomial, that do not require solution of a Vandermonde system. We describe one such in the following:

Suppose $n = 1$, that is, suppose that we have only two points (x_0, f_0), (x_1, f_1), then it is easy to see that the linear polynomial

$$P1(x) = \frac{x - x1}{x0 - x1} f0 + \frac{x - x0}{x1 - x0} f1$$

is an interpolating polynomial, because $P_1(x_0) = f_0$, $P_1(x_1) = f_1$.

For convenience, we shall write the polynomial $P_1(x)$ in the form
$P1(x) = L_0(x)f_0 + L_1(x)f_1$,

Where $Lo(x) = \frac{x - x1}{x0 - x1}$ and $L1(x) = \frac{x - x0}{x1 - x0}$
x

Note that both the polynomials $L_0(x)$ and $L_1(x)$ are polynomials of degree 1.

The concept can be generalized easily for polynomials of higher degrees. To generate polynomials of higher degrees, let's define the set of polynomials $[L_k(x)]$ recursively, as follows:

$$Lk(x) = \frac{(x - x0)(x - x1)........(x - xk - 1)(x - xk + 1)...(x - xn)}{(xk - x0)(xk - x1)....(xk - xk - 1)(xk - xk + 1)....(xk - xn)} \qquad (3)$$

We will now show that the polynomial $P_n(x)$ defined by

$$P_n(x) = L_0(x)f_0 + L_1(x)f_1 + \cdots + L_n(x)f_n \qquad (4)$$

is an interpolating polynomial. To see this, note that

$L_0(x_0) = 1,\ L_0(x_1) = L_0(x_2) = \cdots = L_0(x_n) = 0$
$L_1(x_1) = 1,\ L_1(x_0) = L_1(x_2) = \cdots = L_n(x_n) = 0$

In general
$L_k(x_k) = 1$ and $L_k(x_i) = 0,\ i \neq k$.

Thus

$P_n(x_0) = L_0(x_0)f_0 + L_1(x_0)f_1 + \cdots + L_n(x_0)f_n = f_0$
$P_n(x_1) = L_0(x_1)f_0 + L_1(x_1)f_1 + \cdots + L_n(x_1)f_n = 0 + f_1 + \cdots + 0 = f1$

$P_n(x_n) = L_0(x_n)f_0 + L_1(x_n)f_1 + \cdots + L_n(x_n)f_n = 0 + 0 + \cdots + 0 + f_n = fn$

e.i., the polynomial $P_n(x)$ has the property that

$P_n(x_k) = f_k,\ k = 0, 1, \cdots, n.$

The polynomial $P_n(x)$ defined by (4) is known as the **Lagrange Interpolating Polynomial.**

Example Interpolate f(3) from the following set of data using Lagrange interpolation:

X	0	1	2	4
F(x)	7	13	21	43

Using general formula

$$Lk(x) = \frac{(x-x0)\,(x-x1)........(x-xk-1)(x-xk+1)...(x-xn)}{(xk-x0)\,(xk-x1)....(xk-xk-1)(xk-xk+1)....(xk-xn)}$$

$$L0(x) = \frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)}$$

$$L1(x) = \frac{(x-0)(x-2)(x-4)}{1.(-1)(-3)}$$

$$L2(x) = \frac{(x-0)(x-1)(x-4)}{2 \cdot 1 \cdot (-2)}$$

$$L3(x) = \frac{(x-0)(x-1)(x-2)}{4 \cdot 3 \cdot 2}$$

Thus for x=3

L0(3) = ¼; L1(3) = -1; L2(3) = 3/2; L3(3) = ¼

So, P3(3) = L0(3)x7 + L1(3)x13 + L2(3)x21+L3(3)x43.
= (1/4)x7 + (-1)x13 + (3/2)x21 + (1/4)x 43
= 31

Thus From given set of data the interpolated value of dependent variable corresponding to dependent variable x=3 is f(3) = 31.