Radiation Damping

Lecture 18

1 Introduction to the Abraham-Lorentz equation

Classically, a charged particle radiates energy if it is accelerated. We have previously obtained the Larmor expression for the radiated power as;

$$P = \frac{\mu q^2 c}{6\pi} |\dot{\vec{\beta}}|^2$$

This is written in the MKS system of units and is non-relativistic. However, we have previously noted that this can be placed in a suggestive form since a non-relativistic acceleration equals a force divided by a mass. The force is $\frac{d\vec{p}}{dt}$. Thus we have for the Larmor equation;

$$P = \frac{\mu q^2}{6\pi m^2 c} [\frac{d\vec{p}}{dt} \cdot \frac{d\vec{p}}{dt}]$$

Now write the term in brackets in covarient form;

$$[\frac{d\vec{p}}{dt} \cdot \frac{d\vec{p}}{dt}] \rightarrow [\frac{d\vec{p}^{\mu}}{d\tau} \cdot \frac{d\vec{p}^{\mu}}{d\tau}] = \kappa$$

Then write;

$$\kappa = \gamma^2 [\frac{d\vec{p}}{dt} \cdot \frac{d\vec{p}}{dt} - (\frac{dE}{dt})^2]$$

and use

$$\frac{d\vec{p}}{dt} = \frac{mc^2 \dot{\vec{\beta}}}{\sqrt{1 - \vec{\beta} \cdot \vec{\beta}}} + \frac{mc^2 \vec{\beta}(\vec{\beta} \cdot \dot{\vec{\beta}})}{[1 - \beta^2]^{3/2}}$$

Collecting terms the covarient form for the power is;

$$P = \frac{\mu q^2 c}{6\pi} \gamma^6 [\dot{\beta}^2 - (\vec{\beta} \times \dot{\vec{\beta}})]$$

The above equation is the energy loss due to radiation, and must be accounted in the mechanical motion of charges. The impact on an equation of motion is the addition of a force, F_{rad}, that acts to reduce the mechanical energy of the system. In the above case this is a power loss expressd by $\vec{F}_{rad} \cdot \vec{V}$, where V is the particle velocity. In previous studies, we assumed that this term is small and changes to the velocity can be neglected. Thus in approximation, the motion is calculated without consideration of the energy loss, and then using this trajectory, the energy loss is calculated. The concept is similar to a perturbation expansion with the exception that calculation of terms beyond 1^{st} order are not straightforward.

In the following we want to take into account radiation damping in the equation of motion. We do this non-relativistically, expecting an equation of the form;

$$F_{ext} \;=\; F_{inertial} \,+\, F_{rad}$$

In the above F_{ext} is an externally applied force, $F_{inertial}$ is the inertial force $m\dfrac{dV}{dt}$, and R_{rad} is the radiative damping force. We want to equate the energy going into the EM field to the energy expended in overcoming F_{rad}. We develop this on time average (Gaussian units);

$$Energy \;=\; \int_{t_1}^{t_2} dt\,(\vec{F}_{rad}\cdot\vec{V}) \;=\; (2/3)(q^2/c^3)\int_{t_1}^{t_2} dt\,\dot{V}^2$$

Now integrate the right hand side of the above equation by parts.

$$\int_{t_1}^{t_2} dt\,(\vec{F}_{rad}\cdot\vec{V}) \;=\; (2/3)(q^2/c^3)\,[\dot{\vec{V}}\cdot\vec{V}|_{t_1}^{t_2} \,-\, \int_{t_1}^{t_2} dt\,\ddot{V}\cdot\vec{V}$$

Suppose we have motion such that $\dot{\vec{V}}\cdot\vec{V}|_{t_1}^{t_2} \;=\; 0$, for example periodic motion. Then compare the terms of the integrand in the equation to identify;

$$\vec{F}_{rad} \;=\; -(2/3)(q^2/c^2)\ddot{\vec{V}} \;=\; -m\tau\ddot{\vec{V}}$$

Here we have defined $\tau \;=\; \dfrac{2q^2}{3mc^3}$ (or $\dfrac{\mu q^2}{6\pi mc}$ in the MKS system). When evaluated, τ is very small, for an electron τ is on the order of the Planck time 10^{-23} s. Collecting terms we obtain the Abraham-Lorentz equation;

$$m(\dot{\vec{V}} \,-\, \tau\ddot{\vec{V}}) \;=\; \vec{F}_{ext}$$

2 Solution to the Abraham-Lorentz equation

The Abraham-Lorentz equation has a number of conceptual problems. In the first place the radiative force is proportional to the derivative of the acceleration, so the equation is not Galilean invarient, even though it is a non-relativistic equation of motion. Of course, the equation incorporates EM emergy loss and EM waves are Lorentz not Galilean invarient. Also the solution requires knowledge of an initial position, initial velocity, AND an initial acceleration. However, this later objection can be removed if the initial acceleration is related to initial field. Recall that the derivation of this expression requires periodic motion. Assuming a harmonic solution, odd derivatives of position create a phase difference of $\pi/2$ or

$3\pi/2$ between the force and the position. This phase difference allows the force to act on the movement of the charge to input the required energy. This is identical to the introduction of a frictional force proportional to velocity in the equations of motion, which incorporates Joule heating.

The Abraham-Lorentz equation is

$$m(\dot{\vec{V}} - \tau\ddot{\vec{V}}) = \vec{F}_{ext}$$

For an external force $F_{ext} = 0$ the solution is either $\dot{\vec{V}} = 0$ or $\vec{a}_0 e^{t/\tau}$, where a_0 is the acceleration when $t = 0$. The second solution is non-physical because we have required $\dot{\vec{V}} \cdot \vec{V}|_{t_1}^{t_2} = 0$ and the acceleration would increase without bounds as $\lim_{x\to\infty}$

Now look for more interesting solutions. We first make a change of variable;

$$\dot{\vec{V}} = e^{t/\tau}\,\vec{u}(t)$$

Here $\vec{u}(t)$ is a function of t to be determined. This gives

$$\ddot{\vec{V}} = (e^{t/\tau}/\tau)\vec{u} + e^{t/\tau}\,\dot{\vec{u}}$$

Substitution into the Abraham-Lorentz equation;

$$\dot{\vec{u}} = -(e^{t/\tau}/m\tau)\,\vec{F}_{ext}$$

Integrating we obtain

$$\vec{u} = (1/m\tau)\int_t^{t_0} dt'\,e^{-t'/\tau}\,F(t')$$

and

$$\dot{\vec{V}} = (e^{t/\tau}/m\tau)\int_t^{t_0} dt'\,e^{-t'/\tau}\,F(t')$$

For a constant force $F_{ext} = F_0$

$$\dot{\vec{V}} = F_0/m\,[1 - e^{(t-t_0)/\tau}]$$

When $t = 0$ the acceleration is approximately F/m as expected. However, for a constant force the acceleration is NOT constant, as will be seen in more detail later.

3 Scattering and Absorption of radiation by an oscillator

Choose to model a non-relativisitc oscillator, so the solution has harmonic form and we expect that the Abraham-Lorentz equation can be applied. For non-relativistic motion we write the mechanical equation of motion in the form;

$$m(\ddot{\vec{x}} - \tau\dddot{\vec{x}} + \omega^2\vec{x}) = \vec{F}_{ext}(t)$$

The 1^{st} term in the above is the usual inertial term. The 2^{nd} is the radiative force obtained from the Abraham-Lorentz equation. The 3^{rd} term is the restoring force. There is an applied external force to supply the power. One could also add a term that is proportional to velocity ($\gamma\dot{x}$) which would contribute a dissipative force due to Joule heating (resistance for example). Assume that the applied force is due to the harmonic E field of an EM wave, $(q/m)E_0 e^{-i\omega t}\hat{x}$. We find that the steady state solution is

$$\vec{x} = (q/m)\,\frac{E_0\,e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\omega\gamma - i\omega^2\tau}\,\hat{x}$$

then define a total width $\Gamma = \gamma + \omega\tau$. Substitution gives the equation for the power supplied by the force. To separate the power into dissipative and intertial terms write that $x = x_0\,cos(\omega t + \phi)$ and substitute into the above equation for the force.

$$\vec{F}_{ext} = -m[(\omega^2 - \omega_0^2)x_0\,cos(\omega t + \phi) - \tau\omega^3 x_0\,sin(\omega t + \phi)]$$

Then the power expended by the force is

$$P = \vec{F}_{ext}\cdot\dot{\vec{x}}$$

Suppose we consider the energy supplied over a period.

$$\langle Energy\rangle = \int_0^{2\pi\omega} dt\,\vec{F}_{ext}\cdot\dot{\vec{x}}$$

$$\langle Energy\rangle = m\omega(\omega^2 - \omega_0^2)\int_0^{2\pi\omega} dt\,cos(\omega t + \phi)\,sin(\omega t + \phi) +$$

$$(m\tau\omega^4)\int_0^{2\pi\omega} dt\,sin^2(\omega + \phi)$$

The first term averages to zero, the second to 1/2. The second term represents energy loss to the system. The first represents energy taken from the force over half a cycle and then returned over the 2^{nd} half of the cycle. Note as previously discussed, radiation loss requires a force component that has an odd derivative of the position to be in phase with the velocity. Now substitute for $|x|^2$ to obtain for the energy loss

$$\langle Energy \rangle = \frac{(m\tau\omega^2)^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$

We arrive at the same expression if we use the radiation field

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon c}(1/r)[\hat{n} \times \hat{n} \times \ddot{\vec{x}}]$$

Obtain the power radiated using the Poynting vector $S = (1/\mu c)|\vec{E} \times \vec{B}|$. The cross section for scattering is obtained by the power into a solid angle divided by the incident flux. This yields for a plane polarized wave in the $\hat{x}$ direction;

$$\frac{dP}{d\Omega} = (\frac{q^2}{4\pi\epsilon mc})^2|\hat{\epsilon} \cdot \hat{x}|[\frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$

The shape of the cross section is Lorentzian and is shown in fig. 1. The maximum is located ar $\omega = \omega_0$ and the width of the excitation at the half value points of the peak is Γ. When the energy loss terms due to radiation and resistance go to zero ($\Gamma \rightarrow 0$) the width of the peak $\rightarrow 0$. The factor Γ is inversely proportional to the quality factor, Q, of the system.

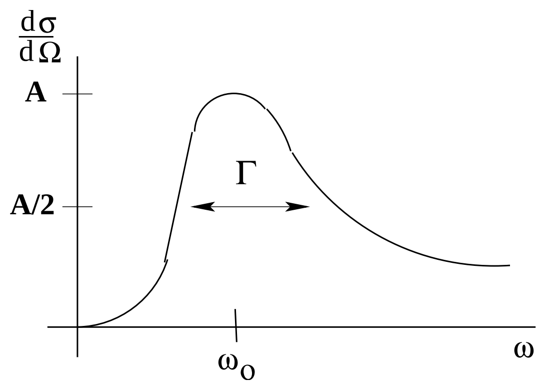

Figure 1: The shape of the resonant cross section (power radiated) for EM scattering from a charged harmonic oscillator

4 Motion in a central potential

Suppose a particle moving in a central potential. The particle will have a trajectory which can be calculated by Newtonian dynamics if we neglect radiation damping. The trajectory can then be used to determine the radiation using the Larmor equations, for example. Suppose a potential, $U(r)$. The force field is then, $\vec{F} = -\vec{\nabla} U = -\frac{dU}{dr}\hat{r}$. Newtonian mechanics yields;

$$\dot{\vec{V}} = -(1/m)\,\frac{dU}{dr}\,\hat{r}$$

The power radiated is then obtained from the Larmor equation;

$$\frac{\mathrm{W}}{dt} = -(\frac{2q^2}{3c^3})\,|\dot{V}|^2 = -(\frac{2q^2}{3c^3})[(1/m^2)(\frac{dU}{dr})^2]$$

In the following we use $\tau = (2/3)\dfrac{q^2}{mc^3}$

However, not only energy is lost in the motion, but angular momentum as well. The time derivative of the angular momentum, $\vec{L} = m(\vec{r} \times \vec{V})$, is

$$\frac{d\vec{L}}{dt} = m(\vec{v} \times \vec{V}) + m(\vec{r} \times \dot{\vec{V}})$$

Now insert the value of $\dot{\vec{V}}$ obtained from the Abraham-Lorentz equation $\vec{F}_{ext} = m\vec{a} - m\tau\dot{\vec{a}}$. Thus

$$\frac{d\vec{L}}{dt} = m(\vec{r} \times \dot{\vec{V}}) = m\tau(\vec{r} \times \ddot{\vec{V}})$$

Take the second time derivative of the angular momentum, and assume that this is nearly zero. Indeed, angular momentum is conserved without radiation, and radiation only slightly reduces its value. Therefore;

$$\frac{d^2\vec{L}}{dt^2} = m(\vec{V} \times \dot{\vec{V}}) + m(\vec{r} \times \ddot{\vec{V}}) \approx 0$$

Then as $\vec{F}/m = \dot{\vec{V}} = -(1/m)(\frac{dU}{dr})\,\hat{r}$ where $U(r)$ is the central potential. We finally obtain;

$$\frac{d\vec{L}}{dt} = -(\tau/m)\langle(1/r)\,\frac{dU}{dr}\rangle\,\vec{L}$$

The above equation gives the change in angular momenum due to radiation.

5 The Collision of 2 charged particles

Consider the motion of 2 charged particles, q_1 and q_2. The positions of these charges are shown in the figure 2. We develop the radiation in the CM frame. In this frame the radiation energy loss field from each particle is proportional to $Q^2|\ddot{\vec{r}} \cdot \ddot{\vec{r}}|$. Thus the total radiation is proportional to the square of;

$$Q\vec{r} = q_1\vec{r}_1 + q_2\vec{r}_2 = [\frac{m_2}{m_1 + m_1}q_2 - \frac{m_2}{m_1 + m_2}q_1]\vec{r}$$

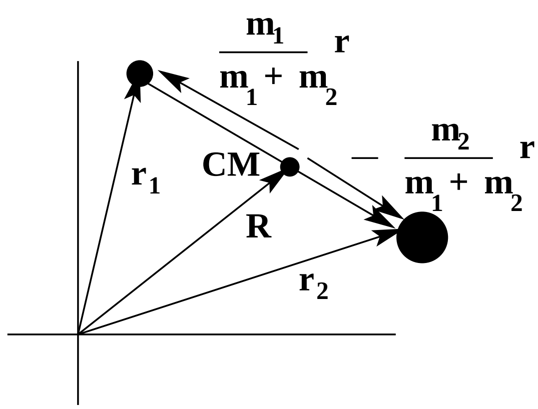

Figure 2: The geometry for a collision of 2 particles. CM is the center of mass point

In the above we have used the relation between the relative vector $\vec{r}$ and the vectors $\vec{r}_1$ and $\vec{r}_2$. We remove the time derivative of the constant CM position vector, R.

$$\vec{r}_1 = \vec{R} + \frac{m2}{m_1 + m_2}\vec{r}$$

$$\vec{r}_2 = \vec{R} - \frac{m1}{m_1 + m_2}\vec{r}$$

In terms of the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$$\vec{d} = q_1\vec{r}_1 + q_2\vec{r}_2 = \mu(q_1/m_1 - q_2/m2)\vec{r}$$

Here $\vec{d}$ is the dipole moment. Now we assume that $d \rightarrow de^{-i\omega t}$ and insert this into the Larmor equation for the radiated power;

$$Power = \frac{2}{3c^3}|\ddot{d}|^2 = \omega^4 \frac{2\omega^4}{3c^3}|d|^2$$

On the other hand if we know the trajectory of the particles, we could take the time derivative to get $\ddot{\vec{d}}$.

6 Rutherford scattering

To complete the above description of radiation when charged particles scatter, we require the trajectories of the charges as they move subject to the Coulomb potential. This is called

Rutherford scattering, although it also describes bound state motion, and in fact motion of any 2 particles interacting via a $1/r$ potential, *eg* gravitational attraction. The geometry is shown in figure 3. In this case we ignore the motion of the heavier particle, or we use CM coordinates.

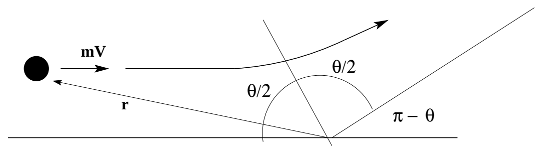

Figure 3: Motion of a particle in a central $(1/r)$ potnetial.

To apply Newtonian mechanics we need the acceleration of the particle. The solution is sought in spherical coordinates. Because of azmuthal symmetry we use polar coordinates with variables (r, θ). Then ;

$$\vec{r} \,=\, r\hat{r}$$

$$\vec{V} \,=\, \frac{dr}{dt}\hat{r} + r\frac{\hat{r}}{dt} \,=\, V_r\hat{r} \,+\, r\omega\hat{\theta}$$

$$\vec{a} \,=\, \frac{dV_r}{dt}\hat{r} + V_r\frac{\hat{r}}{dt} \,+\, \omega\frac{dr}{dt}\hat{\theta} \,+\, r\hat{\theta}\frac{\omega}{dt} \,+\, r\omega\frac{\hat{\omega}}{dt}$$

$$\vec{a} \,=\, (a_r - r\omega^2)\hat{r} \,+\, (\alpha r + 2V\omega)\hat{\theta}$$

Here $\alpha = \dfrac{d\omega}{dt}$. Then we have;

$$m\vec{a} \,=\, (qQ/r^2)\hat{r}$$

This means;

$$(\alpha r + 2V\omega) \,=\, 0$$

with solution $\dfrac{dr^2\omega}{dt} \,=\, L$ where L is a constant and is the system angular monentum. We can use this to convert the time derivative to one with respect to θ.

$$\frac{d}{dt} \,=\, \frac{L}{mr^2}\,\frac{d}{d\theta}.$$

We integrate the above second order equation of motion, or better start from energy conservation,

$$(1/2)m\dot{r}^2 \,+\, (1/2)(L^2/mr^2) \,+ qQ/r \,=\, Constant \,=\, InitialEnergy(E_0)$$

The second term is the rotational energy $(1/2)I\omega^2$ where I is the moment of inertia, and the $3^r d$ term is the potential energy, $U(r)$. We transform this to a differential equation in terms of θ and r.

$$d\theta \,=\, \frac{Ldr}{mr^2}\sqrt{(2/m)(E_0 - U(r) - L^2/2mr^2)}$$

The solution obtained by integration is

$$(1/r) \,=\, -\frac{mqQ}{L^2}(1 \,+\, \epsilon\, cos(\theta/2))$$

This is the equation of a hyperbola or an ellipse depending on the sign of the energy. The excentricity, ϵ, is

$$\epsilon \,=\, \sqrt{1 + \frac{2E_0L^2}{m(qQ)^2}}$$

The velocity and acceleration of the charge q can be found using the energy equation above to get the radiated power.

$$\frac{d}{dt} \,=\, \frac{d\theta}{dt}\,\frac{d}{d\theta}$$

7 1-D motion in the Abraham-Lorentz model

In the Abraham-Lorentz model the equation of motion in 1-D is;

$$m(a - (1/b)\dot{a}) \,=\, f(t)_{ext}$$

Here we have defined $1/b = \tau \,=\, (2q^2/3mc^3)$ and we have noted that τ is an extremely small time. The externally applied force, $f(t)_{ext}$, is now written, $f(t)$. If $1/b$ is neglected then Newton's equation $f = ma$ results. In this case, integration provides the velocity and position in order to compare with solutions when $(1/b)$ is not neglected.

$$V \,=\, V_0 \,+\, (1/m)\int_0^t dt'\, f(t')$$

$$x \,=\, x_0 \,+\, V_0 t\,(1/m)\int_0^t dt'\,(t - t')\, f(t')$$

Note that integration is up to the present time, t. The boundry conditions require the specification of x_0 and V_0 at $t = 0$. Now if radiation is allowed;

$$ma \;-\; m\dot{a}/b \;=\; f(t)$$

The solution for the acceleration is;

$$a(t) \;=\; e^{bt}[a_0 \;-\; (b/m) \int_0^t dt'\, e^{-bt'} \; f(t')]$$

Here there is the additional boundry condition that requires specification of a_0 at $t = 0$. For an arbitrary a_0, the particle acceleration increases without bounds as $t \to \infty$. To prevent this we choose;

$$a_0 \;=\; (b/m) \int_0^\infty dt'\, e^{-bt'} \; f(t')$$

This means that;

$$a(t) \;=\; (b/m) \int_t^\infty dt'\, e^{-b(t'-t)} \; f(t')$$

Here $a(t) \to 0$ as $t \to \infty$. We rewrite this equation using $\tau = (t' - t)$ and completing the integration to get v and x.

$$a(t) \;+\; (b/m) \int_0^\infty d\tau\, e^{-b\tau} f(t + \tau)$$

$$v(t) \;=\; V_0 \;+\; (1/m) \int_0^t dt'\, f(t') \;+\; (1/m) \int_0^\infty dt'\, e^{-bt'} [f(t + t') - f(t')]$$

$$x(t) \;=\; x_0 + V_0 t + (1/m) \int_0^t dt'\, (t - t') f(t') + (1/bm) \int_0^t dt'\, f(t') + (1/bm) \int_t^\infty dt'\, e^{-b(t'-t)} f(t') - (1 + bt)/bm \int_0^\infty dt'\, e^{-bt'} \; f(t')$$

There is no divegence in these equations but there are non-local effects which conflict with the concept of causality, as will be discussed in the next lecture.